Dayahead Electricity Pricing for a Heterogeneous Microgrid under Arbitrary Utility and Cost Structures

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Abstract—**In this paper, we consider the dayahead load and supply allocation problem in an electrical microgrid which consists of a system manager, consumers, and suppliers. We do not impose any mathematical, operational, and economic assumptions on the consumers and suppliers other than restricting them to have finite number of load and supply schedules. For example, a consumer can have arbitrary forms of flexibilities in its demand, such as deferrability or intermittence, and the user can choose from a set of predetermined load allocations that represent these flexibilities. Thus, our system model can accommodate heterogeneous user requirements and constraints in the same problem formulation. Under this system model, we formulate a welfare maximization problem, and derive a distributed primal-dual pricing algorithm. In our algorithm, the system manager generates prices for the load and supply allocations of the users, and given these prices the users independently choose the schedules to their best operational and economic interests. Furthermore, the pricing algorithm incentivizes the users so that their decisions benefit the system performance, and it achieves the optimal load and supply allocation. Finally, we study the algorithm's performance via simulations, and demonstrate how the demand-side flexibilities are utilized for the system's benefit.**

I. INTRODUCTION

THe electrical grid is a complex heterogeneous networked system that is composed of entities with quite different characteristics in both supply and demand sides. Smart grid, towards which the electrical grid is evolving, encompasses even more heterogeneity with the emerging concepts such as distributed generation, renewable supply, and demand flexibility [1]. In this work, we design pricing and resource allocation techniques for a smart electrical grid where demand and supply side entities have arbitrary and heterogeneous requirements.

We consider a microgrid that comprises a system manager, consumers, and suppliers. The system manager aggregates its subscribers' demand, and serves their load by procuring electricity from suppliers. The system manager's goal is to determine energy prices that coordinate demand and supply toward efficient social welfare. The system manager can be thought of as a Load Aggregator (LA) [2], and we are particularly interested in the LA's dayahead load and supply allocation problem. The LA can manage a broad range of large-scale consumers such as data centers, supermarkets, and factories [3], [4]. On the other hand, suppliers can be local power plants, renewable generation companies, or retailers which have quite different operational and economic constraints [5], [6]. Overall, we consider a system in which both supply and demand sides are composed of participants with arbitrary and heterogeneous characteristics.

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The load and supply management, and pricing problem has been extensively studied in both traditional and smart electrical grid contexts under various cost, utility, and demand flexibility models. For example, convex cost functions are used to model generation costs [7]–[10], cost of procurement from wholesale markets [11], load shedding cost for demand response [12], distribution costs [13], storage costs [14]. Moreover, concave utility functions are employed to model consumer satisfaction [7], [10]–[12], [14], and disutilities associated with service delays [15], [16]. On the other hand, flexibilities in consumer demand are modeled by shifting load in time [16]–[19], deferring the time of service [15], [20]–[23], setting maximum and minimum consumption constraints over a finite period of time [24], [25], enforcing average consumption goals [26]. Furthermore, under the aforementioned models and constraints, pricing strategies are employed in the literature to incentivize demand and supply toward system manager's goals: [7], [9]–[11], [17] implement optimization based marginal pricing, [12], [13] derive prices based on competitive equilibrium and game theoretic analysis. We note that despite providing tractable problem formulations and solutions, these models and constraints restrict the system model and its participants to have quite specific structures and features.

In this paper, we are interested in the LA's dayahead load and supply allocation problem under arbitrary cost, utility, and flexibility models. In our problem formulation, we do not assume any particular structure for the consumer utility and supplier cost functions. For example, consumers can choose arbitrary mappings that assign usage to utility values. On the supplier side, generators can incorporate unit start up costs and ramp constraints into their supply cost structures, or retailers can simply adopt their wholesale market payments as their supply cost. Furthermore, demand flexibilities can take various forms to represent heterogeneous consumer models in the same problem formulation. For instance, some of the users can shift their demand in time or simply delay their consumption, some of the users may only need to adjust the amount of their load, and some other users can choose between intermittent and consecutive service.

In particular, we formulate a social welfare maximization problem for the LA. Due to the arbitrary cost and utility structures and heterogeneous demand flexibilities, the welfare maximization problem does not have a closed form solution. Instead, by a series of transformations, we convert the optimization problem into a linear problem that can be solved by iterative and distributed algorithms. Then, we design a primal-dual algorithm which assigns prices to consumers' and suppliers' load allocations. The generated prices incentivize consumers and suppliers to allocate their load and supply, respectively, so that the aggregate load and supply allocation benefits the whole system performance.

The rest of the paper is organized as follows: In Section II-A, we present the system model and describe its participants in detail. In Section II-B, we formulate the social welfare maximization problem of the LA. Then, we transform the problem into a linear program by introducing auxiliary variables, and feasible consumption and supply schedules. In Section III, the dayahead pricing algorithm, which achieves the optimal load and supply allocation, is derived. The algorithm is distributed and assumes that the system participants independently act to their own benefits. In Section IV, we study the algorithm's performance via numerical investigations. Simulations show that our algorithm assign prices to consumers' and suppliers' schedules so that they are incentivized to choose the schedules that benefit the system performance.

II. System Model and Problem Formulation

In this section, we present the system model and introduce the participants of the system: the LA, consumers, and suppliers. Then, we formulate the demand and supply management problem of the LA as a welfare maximization problem.

A. System Model

We consider an LA whose aim is to serve its consumers' daily energy demand by providing electricity from suppliers. The LA manages its consumers' demand and decides on their daily loads, and procures the matching amount of supply one day ahead. A day is divided into T control slots, thus the LA is responsible for determining T-hour load and supply schedules for the consumers and suppliers for each day, one day before the load and supply are actually realized.

Consumers can be factories with industrial loads who are willing to alter the time and amount of their energy consumption. Hence, the consumers' demand is said to be *flexible*. On the other hand, suppliers can be local power plants, or retailers who are willing to procure electricity. In the following, we describe the system participants in more detail.

1. Load Aggregator (LA): The LA aggregates its consumers' demand, and serves the total load by procuring electricity from the suppliers [2]. The LA is responsible for the benefits of all parties involved in the system. In particular, the objective of the LA is to maximize the social welfare which is defined as the total utility obtained by consumers for using electricity minus the total cost incurred on suppliers for procuring electricity.

2. Suppliers: There are M suppliers that interact with the LA. Each supplier has a finite number of *supply schedules* which determine the amount of procurement at each slot throughout a day. A supply schedule of supplier m is denoted by $\mathbf{s}^m$, which is a length T vector. The set of all feasible schedules of supplier m is denoted by $\mathcal{S}^m$. The feasible schedules chosen by a supplier take into account its operational constraints. Supplying the schedule $\mathbf{s}^m$ incurs the cost $C^m(\mathbf{s}^m)$. We do not assume any structure for C^m such as convexity or differentiability; C^m is an *arbitrary* mapping from $\mathcal{S}^m$ to $\mathbb{R}$. For instance, it can include the unit start up costs of a generator, or it can represent wholesale market purchases of a retailer. The objective of each supplier is to maximize its profit that is defined as the payment it receives minus the cost of procurement.

3. Consumers: There are N consumers that interact with the LA. Consumers have flexible demand, of which time and consumption amount can be altered. Each consumer has a finite number of *consumption schedules* which is composed of the possible realizations of load over T time slots for the flexible demand. Thus, a consumer's choice of its feasible schedules encompasses the flexibilities that its demand bears, such as shiftability, deferrability, and intermittence. A consumption schedule of consumer n is denoted by $\mathbf{x}^n$, which is a length T vector. The set of all schedules of consumer n is denoted by $\mathcal{X}^n$. By consuming the load $\mathbf{x}^n$, consumer n obtains the utility $U^n(\mathbf{x}^n)$. We do not assume any structure for U^n; U^n is an *arbitrary* mapping from $\mathcal{X}^n$ to $\mathbb{R}$. The objective of each consumer is to maximize its welfare which is defined as the utility it obtains from consumption minus the payment it makes for the consumption.

Under the described system model, both consumers and suppliers determine their load and supply schedules, utility and cost functions, before the LA generates prices. Users can employ prediction methods in the process of determining their schedules and corresponding utility and cost functions, but this process is out of this paper's scope.

B. Problem Formulation

We formulate the social welfare maximization problem of the LA as follows:

$$\max_{\{\mathbf{x}^n\},\{\mathbf{s}^m\}} \quad \sum_{n=1}^{N} U^n(\mathbf{x}^n) - \sum_{m=1}^{M} C^m(\mathbf{s}^m) \tag{1}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_t^n = \sum_{m=1}^{M} s_t^m, \quad \forall t \in \{1,\ldots,T\} \tag{2}$$

$$\mathbf{x}^n \in \mathcal{X}^n,\ \forall n$$

$$\mathbf{s}^m \in \mathcal{S}^m,\ \forall m.$$

where (2) is the load-supply matching constraint. The optimization is over the discrete and finite feasible schedule sets of consumers and suppliers, $\mathcal{X}^n$ and $\mathcal{S}^m$ respectively, which encompass the operational and economic constraints of the users. In order to guarantee the existence of a feasible solution that satisfies (2), we assume that each user's set contains the schedule which have 0 at all slots t, i.e. $\mathbf{x}^n = \mathbf{0}$.

The problem in (1) can be reformulated in a compact form by including the constraint (2) in the following set definition.

$$\mathcal{F} \triangleq \Bigg\{ (\mathbf{x},\mathbf{s}) : \mathbf{x} = (\mathbf{x}^n : \mathbf{x}^n \in \mathcal{X}^n),\, \mathbf{s} = (\mathbf{s}^m : \mathbf{s}^m \in \mathcal{S}^m), \sum_{n} x_t^n = \sum_{m} s_t^m,\, \forall n,m,t \Bigg\}.$$

Then, the problem (1) becomes

$$\max_{\mathcal{F}} \quad \sum_{n=1}^{N} U^n(\mathbf{x}^n) - \sum_{m=1}^{M} C^m(\mathbf{s}^m) \tag{3}$$

Problem (1) is a nonlinear optimization problem over a discrete and finite set, and it does not have any particular structure such as convexity or linearity due to the arbitrary choice of utility and cost functions. Furthermore, the knowledge of the utility and cost functions is private to consumers and suppliers, and hence it is practically not available to the LA. Consequently, due to the complexity of the problem and the privacy of the utility and cost functions, it is not possible for the LA to solve problem (1) centrally. Therefore, we are interested in distributed and possibly iterative solutions to problem (1).

Now, consider the following linear problem which is equivalent to (3):

$$\begin{aligned} \min_{\gamma} \quad & \gamma \\ \text{s.t.} \quad & \gamma \geq \sum_{n=1}^{N} U^n(\mathbf{x}^n) - \sum_{m=1}^{M} C^m(\mathbf{s}^m),\ \forall (\mathbf{x},\mathbf{s}) \in \mathcal{F} \end{aligned} \tag{4}$$

Observe that at the optimal solution of the above problem, γ is equal to the maximum objective value of the original welfare maximization problem in (1).

Next, we define $\pi_x^n \geq U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$ and $\pi_s^m \geq p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m)$ where π_x^n and π_s^m are the surplus terms for consumer n and supplier m, respectively, and $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$ are the *bundle prices* communicated to consumer n for schedule $\mathbf{x}^n$ and to supplier m for schedule $\mathbf{s}^m$, respectively. We note that prices $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$ are assigned to schedules of users; they do not represent the price of energy at a certain time slot. As described in Section II-A, a schedule is a *bundle* of consumption or supply amounts for all time slots from $t = 1$ to $t = T$. Thus, we refer to these prices, that are assigned to schedules, as *bundle prices*. Consequently, the price of energy at a time slot is not defined under this formulation.

Define $\Phi \triangleq \{\gamma, \pi_x^n, \pi_s^m, p^n(\mathbf{x}^n), p^m(\mathbf{s}^m)\}$ as the set of problem variables. Using the surplus terms and bundle prices, we rewrite problem (4) as

$$\begin{aligned} \min_{\Phi} \quad & \gamma \\ \text{s.t.} \quad & \gamma \geq \sum_{n=1}^{N} \left(\pi_x^n + p^n(\mathbf{x}^n)\right) - \sum_{m=1}^{M} \left(p^m(\mathbf{s}^m) - \pi_s^m\right), \\ & \qquad \forall (\mathbf{x},\mathbf{s}) \in \mathcal{F} \\ & \pi_x^n \geq U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n),\ \forall n, \mathbf{x}^n \in \mathcal{X}^n \\ & \pi_s^m \geq p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m),\ \forall m, \mathbf{s}^m \in \mathcal{S}^m \end{aligned} \tag{5}$$

We further simplify the constraint on γ by defining $\pi \geq \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m)$:

$$\begin{aligned} \min_{\Phi} \quad & \gamma \\ \text{s.t.} \quad & \gamma \geq \pi + \sum_{n=1}^{N} \pi_x^n + \sum_{m=1}^{M} \pi_s^m \\ & \pi \geq \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m),\ \forall (\mathbf{x},\mathbf{s}) \in \mathcal{F} \\ & \pi_x^n \geq U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n),\ \forall n, \mathbf{x}^n \in \mathcal{X}^n \\ & \pi_s^m \geq p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m),\ \forall m, \mathbf{s}^m \in \mathcal{S}^m \end{aligned} \tag{6}$$

where the set Φ includes the newly defined variable π as well. Finally, by dropping γ we obtain

$$\min_{\Phi} \quad \pi + \sum_{n=1}^{N} \pi_x^n + \sum_{m=1}^{M} \pi_s^m \tag{7}$$

$$\text{s.t.} \quad \pi \geq \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m),\ \forall (\mathbf{x},\mathbf{s}) \in \mathcal{F} \tag{8}$$

$$\pi_x^n \geq U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n),\ \forall n, \mathbf{x}^n \in \mathcal{X}^n \tag{9}$$

$$\pi_s^m \geq p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m),\ \forall m, \mathbf{s}^m \in \mathcal{S}^m \tag{10}$$

where Φ is redefined as $\Phi \triangleq \{\pi, \pi_x^n, \pi_s^m, p^n(\mathbf{x}^n), p^m(\mathbf{s}^m)\}$.

III. Optimal Bundle Pricing

Note that problem (7) is a linear program over the set Φ. However, it is still difficult to be solved by the LA, because U^n and C^m are arbitrary functions, and also their knowledge is not available to the LA. Thus, in order to obtain an algorithm that can efficiently solve the optimization problem without requiring the knowledge of the utility and cost functions, we apply primal-dual decomposition techniques [27]. To that end, we first obtain the dual problem of (7) as follows:

$$\max_{\lambda,\mu^n,\mu^m} \quad \sum_n \sum_{\mathcal{X}^n} \mu^n(\mathbf{x}^n) U^n(\mathbf{x}^n) - \sum_m \sum_{\mathcal{S}^m} \mu^m(\mathbf{s}^m) C^m(\mathbf{s}^m) \tag{11}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{\mathcal{F}} \lambda(\mathbf{x},\mathbf{s}) = 1 \\ & \sum_{\mathcal{X}^n} \mu^n(\mathbf{x}^n) = 1,\ \forall n \\ & \sum_{\mathcal{S}^m} \mu^m(\mathbf{s}^m) = 1,\ \forall m \\ & \mu^n(\mathbf{x}^n) = \sum_{\substack{\mathcal{F} \\ \text{s.t. } \mathbf{x}^n \in \mathbf{x}}} \lambda(\mathbf{x},\mathbf{s}),\ \forall n, \mathbf{x}^n \in \mathcal{X}^n \\ & \mu^m(\mathbf{s}^m) = \sum_{\substack{\mathcal{F} \\ \text{s.t. } \mathbf{s}^m \in \mathbf{s}}} \lambda(\mathbf{x},\mathbf{s}),\ \forall m, \mathbf{s}^m \in \mathcal{S}^m \\ & \lambda(\mathbf{x},\mathbf{s}), \mu^n(\mathbf{x}^n), \mu^m(\mathbf{s}^m) \geq 0 \end{aligned}$$

where $\lambda(\mathbf{x},\mathbf{s})$, $\mu^n(\mathbf{x}^n)$, $\mu^m(\mathbf{s}^m)$ are the dual variables corresponding to the constraints in (7), respectively.

We observe that the dual problem (11) coincides with the standard primal problems that are usually formulated in combinatorial auction settings [28]. Hence, the primal problem (7) resembles the dual of a combinatorial auction problem. However, we begin designing the pricing mechanism with (7) as it explicitly states the prices that are of primary interest.

The dual problem (11) is linear in the dual variables, so any feasible primal-dual pair is optimal if it satisfies the corresponding complementary slackness (CS) conditions. Thus, given a primal-feasible solution of (7), the set of equations derived from the CS conditions can be solved for dual-feasible variables. If such dual variables exist, they constitute the primal-dual optimal pair with the given primal variables. Otherwise, a restricted primal problem and its dual are constructed for the CS conditions. The restricted problem measures the amount of violation of the CS conditions, and *it* *provides feasible primal-dual update directions which strictly improve the objective value and guarantee convergence to an optimal solution* [27].

CS conditions for (7) and (11) are as follows:

$$\begin{aligned}
&\lambda(\mathbf{x},\mathbf{s}) = 0, \text{ for } (\mathbf{x},\mathbf{s}) \in \mathcal{F} \\
&\qquad \text{s.t. } \pi > \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m) \qquad (12) \\
&\mu^n(\mathbf{x}^n) = 0, \text{ for } n, \mathbf{x}^n \in \mathcal{X}^n \\
&\qquad \text{s.t. } \pi_x^n > U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n) \\
&\mu^m(\mathbf{s}^m) = 0, \text{ for } m, \mathbf{s}^m \in \mathcal{S}^m \\
&\qquad \text{s.t. } \pi_s^m > p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m)
\end{aligned}$$

Primal constraints in (7) and Dual constraints in (11)

Restricted problem for (12), which we call the restricted dual problem, is given as

$$\begin{aligned}
\min_{\lambda,\mu,a,b} \quad & a(\pi) + b(\pi) + \sum_n a(\pi_x^n) + b(\pi_x^n) \\
& + \sum_m a(\pi_s^m) + b(\pi_s^m) + \sum_n \sum_{\mathcal{X}^n} a\left(p^n(\mathbf{x}^n)\right) + b\left(p^n(\mathbf{x}^n)\right) \\
& + \sum_m \sum_{\mathcal{S}^m} a\left(p^m(\mathbf{s}^m)\right) + b\left(p^m(\mathbf{s}^m)\right) \qquad (13) \\
\text{s.t.} \quad & \lambda(\mathbf{x},\mathbf{s}) = 0, \quad \text{for } (\mathbf{x},\mathbf{s}) \in \mathcal{F} \\
& \qquad \text{s.t. } \pi > \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m) \\
& \mu^n(\mathbf{x}^n) = 0, \quad \text{for } n, \mathbf{x}^n \in \mathcal{X}^n \\
& \qquad \text{s.t. } \pi_x^n > U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n) \\
& \mu^m(\mathbf{s}^m) = 0, \quad \text{for } m, \mathbf{s}^m \in \mathcal{S}^m \\
& \qquad \text{s.t. } \pi_s^m > p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m) \\
& \sum_{\mathcal{F}} \lambda(\mathbf{x},\mathbf{s}) - a(\pi) + b(\pi) = 1 \\
& \sum_{\mathcal{X}^n} \mu^n(\mathbf{x}^n) - a(\pi_x^n) + b(\pi_x^n) = 1 \\
& \sum_{\mathcal{S}^m} \mu^m(\mathbf{s}^m) - a(\pi_s^m) + b(\pi_s^m) = 1 \\
& \mu^n(\mathbf{x}^n) - a(p^n(\mathbf{x}^n)) + b(p^n(\mathbf{x}^n)) \\
& \qquad = \sum_{\substack{\mathcal{F} \\ \text{s.t. } \mathbf{x}^n \in \mathbf{x}}} \lambda(\mathbf{x},\mathbf{s}), \ \forall n, \mathbf{x}^n \in \mathcal{X}^n \\
& \mu^m(\mathbf{s}^m) - a(p^m(\mathbf{s}^m)) + b(p^m(\mathbf{s}^m)) \\
& \qquad = \sum_{\substack{\mathcal{F} \\ \text{s.t. } \mathbf{s}^m \in \mathbf{s}}} \lambda(\mathbf{x},\mathbf{s}), \ \forall m, \mathbf{s}^m \in \mathcal{S}^m \\
& a(\pi), b(\pi), a(\pi_x^n), b(\pi_x^n), a(\pi_s^m), b(\pi_s^m), \\
& a\left(p^n(\mathbf{x}^n)\right), b\left(p^n(\mathbf{x}^n)\right), a\left(p^m(\mathbf{s}^m)\right), b\left(p^m(\mathbf{s}^m)\right) \geq 0
\end{aligned}$$

where the positive variables a and b measure the violation of the constraints given the primal variables. Thus, objective value of problem (13) is strictly positive if there does not exist a solution of dual variables to the CS equations in (12) given the primal variables. Otherwise, the objective value is 0. Hence, problem (13) can be solved for the dual variables instead of the set of CS equations in (12) for verifying primal-dual optimality. Furthermore, the restricted dual problem together with its primal provide a way to obtain update directions for the primal variables π, π_x^n, π_s^m, $p^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m)$. For that purpose, we write the primal of the restricted dual problem (13) as follows

$$\begin{aligned}
\max_{\bar{\pi},\bar{p}} \quad & -\bar{\pi} - \sum_n \bar{\pi}_x^n - \sum_m \bar{\pi}_s^m \qquad (14) \\
\text{s.t.} \quad & -1 \leq \bar{\pi} \leq 1 \\
& -1 \leq \bar{\pi}_x^n \leq 1, \ \forall n \\
& -1 \leq \bar{\pi}_s^m \leq 1, \ \forall m \\
& -1 \leq \bar{p}^n(\mathbf{x}^n) \leq 1, \ \forall n, \mathbf{x}^n \in \mathcal{X}^n \\
& -1 \leq \bar{p}^m(\mathbf{s}^m) \leq 1, \ \forall m, \mathbf{s}^m \in \mathcal{S}^m
\end{aligned}$$

$$\bar{\pi}_x^n + \bar{p}^n(\mathbf{x}^n) \geq 0, \ \text{ for } \mathbf{x}^n \text{ s.t. } \pi_x^n = U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n) \qquad (15)$$

$$\bar{\pi}_s^m + \bar{p}^m(\mathbf{s}^m) \geq 0, \ \text{ for } \mathbf{s}^m \text{ s.t. } \pi_s^m = p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m) \qquad (16)$$

$$\begin{aligned}
&\bar{\pi} - \sum_n \bar{p}^n(\mathbf{x}^n) - \sum_m \bar{p}^m(\mathbf{s}^m) \geq 0, \\
&\qquad \text{for } (\mathbf{x},\mathbf{s}) \text{ s.t. } \pi = \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m) \qquad (17)
\end{aligned}$$

where $\bar{\pi}$, $\bar{\pi}_x^n$, $\bar{\pi}_s^m$, $\bar{p}^n(\mathbf{x}^n)$, $\bar{p}^m(\mathbf{s}^m)$ are the dual variables corresponding to the constraints in (13). For convenience, the dual variables are named after the variables specifying the condition in each constraint equation, e.g. $\bar{\pi}$ is the dual variable corresponding to the fourth constraint in (13).

The update directions for the primal variables are obtained by solving the restricted primal problem in (14). In particular, the primal variables π, π_x^n, π_s^m, $p^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m)$ are updated, respectively, by appropriately scaled values of $\bar{\pi}$, $\bar{\pi}_x^n$, $\bar{\pi}_s^m$, $\bar{p}^n(\mathbf{x}^n)$, $\bar{p}^m(\mathbf{s}^m)$ which solve the restricted primal problem (14). The following lemma states the feasibility of such update directions and shows that the objective value of (7) is strictly improved towards its optimal value.

Lemma 1. *Let π, π_x^n, π_s^m, $p^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m)$ be primal-feasible for (7). Given these primal variables suppose there is no solution to the set of equations in (12) (or equivalently the objective value of the restricted dual problem (13) is positive). Also, let $\bar{\pi}$, $\bar{\pi}_x^n$, $\bar{\pi}_s^m$, $\bar{p}^n(\mathbf{x}^n)$, $\bar{p}^m(\mathbf{s}^m)$ be a solution to the restricted primal problem (14). Then, $\pi + \delta\bar{\pi}$, $\pi_x^n + \delta\bar{\pi}_x^n$, $\pi_s^m + \delta\bar{\pi}_s^m$, $p^n(\mathbf{x}^n) + \delta\bar{p}^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m) + \delta\bar{p}^m(\mathbf{s}^m)$ are feasible for (7) for sufficiently small $\delta > 0$. Furthermore, the objective value of (7) strictly decreases with the updated variables.*

Proof. First, we show the feasibility, i.e. the updated variables satisfy the constraints (8)-(10). Here, we consider the proof for the constraint (8); the proofs for the constraints (9), (10) follow the same line of reasoning. If $\pi = \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m)$, then $\bar{\pi} \geq \sum_n \bar{p}^n(\mathbf{x}^n) - \sum_m \bar{p}^m(\mathbf{s}^m)$ from (17). Summing together we obtain $\pi + \delta\bar{\pi} \geq \sum_n (p^n(\mathbf{x}^n) + \delta\bar{p}^n(\mathbf{x}^n)) - \sum_m (p^m(\mathbf{s}^m) + \delta\bar{p}^m(\mathbf{s}^m))$. On the other hand, if $\pi > \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m)$, then for sufficiently small δ we have $\pi \geq \sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m) - \delta\left(\bar{\pi} - \sum_n \bar{p}^n(\mathbf{x}^n) + \sum_m \bar{p}^m(\mathbf{s}^m)\right)$, and $\pi + \delta\bar{\pi} \geq \sum_n (p^n(\mathbf{x}^n) + \delta\bar{p}^n(\mathbf{x}^n)) - \sum_m (p^m(\mathbf{s}^m) + \delta\bar{p}^m(\mathbf{s}^m))$. A a result, constraint (8) is satisfied for the updated variables. Following the same approach, we can show that constraints

(9)-(10) also hold for the updated dual variables for sufficiently small δ.

Second, we show that the objective value of (7) strictly decreases with the updated variables. Observe that strong duality holds for the problems (13) and (14). Also, observe that the objective value of (13) is strictly positive if there is no solution for the CS conditions in (12). Thus, objective value of (14) is also positive. We obtain

$$\begin{aligned}\pi + \sum_n \pi_x^n + \sum_m \pi_s^m &> \pi + \sum_n \pi_x^n + \sum_m \pi_s^m \\ &\quad - \left(-\bar{\pi} - \sum_n \bar{\pi}_x^n - \sum_m \bar{\pi}_s^m \right) \\ &= (\pi + \bar{\pi}) + \sum_n (\pi_x^n + \bar{\pi}_x^n) + \sum_m (\pi_s^m + \bar{\pi}_s^m). \quad (18)\end{aligned}$$

□

Up to this point, we demonstrated how to obtain the optimal primal-dual solution for the problems (7) and (11) by solving the CS equations in (12) or by solving the restricted dual problem in (13). We also provided update directions, as the solution of the restricted primal problem (14), which guarantee feasibility after the update and also ensure an improved objective value that strictly decreases towards the optimal value of (7). However, solving the CS equations or the problems (13) and (14) requires the knowledge of the utility and cost functions U^n and C^m. Furthermore, the arbitrary structure of these functions makes this task quite difficult even if they are known by the LA. Fortunately, we obtain the following lemma characterizing additional properties for the problem constraints and update directions obtained from (14).

Lemma 2. *Let π, π_x^n, π_s^m, $p^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m)$ be primal-feasible for (7). Given these primal variables suppose there is no solution to the set of equations in (12) (or equivalently the objective value of the restricted dual problem in (13) is positive). Also suppose that the following relations hold: $\pi_x^n = \max_{\mathbf{x}^n} U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$, and $\pi_s^m = \max_{\mathbf{s}^m} p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m)$. If the primal variables are updated with an appropriate step size δ using the solution of the restricted primal problem, then the above conditions continue to hold.*

Proof. Let $\hat{\mathbf{x}}^n \in \arg\max_{\mathbf{x}^n} U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$. By definition, $\pi_x^n = U^n(\hat{\mathbf{x}}^n) - p^n(\hat{\mathbf{x}}^n)$. Observe that constraint (15) is binding for $\hat{\mathbf{x}}^n$, because otherwise $\bar{\pi}_x^n$ could be further decreased to obtain a better objective value. Using the expression in (15) as equality we obtain $\pi_x^n + \delta\bar{\pi}_x^n = U^n(\hat{\mathbf{x}}^n) - (p^n(\hat{\mathbf{x}}^n) + \bar{p}^n(\hat{\mathbf{x}}^n))$. Furthermore, for all $\mathbf{x}^n \in \mathcal{X}^n$ other than $\hat{\mathbf{x}}^n$, we have $\pi_x^n > U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$. Then, for sufficiently small step size δ we can obtain $\pi_x^n + \delta\bar{\pi}_x^n \geq U^n(\mathbf{x}^n) - (p^n(\mathbf{x}^n) + \bar{p}^n(\mathbf{x}^n))$. As a result, for sufficiently small δ, we have $\pi_x^n + \delta\bar{\pi}_x^n = \max_{\mathcal{X}^n} U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$, which completes the proof. The same result can be proven for π_s^m in the same manner. □

Lemma 2 suggests that if the conditions mentioned in the lemma are satisfied at one iteration, they can be preserved for the rest of the iterations by requiring the suppliers and consumers to choose their schedules to maximize their own benefits given the bundle prices $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$. By doing so, at each step of the primal-dual update process, we will know that the constraints in (9) and (10) are binding only for the schedules that are chosen by the suppliers and consumers. Hence, checking the CS conditions will be trivial for the rest of the elements of the feasible schedule set, because the corresponding dual variables will be zero. Bringing together the observations on Lemma 1 and 2, we propose the distributed primal-dual algorithm in Algorithm BP.

Algorithm BP Bundle Pricing Algorithm

At iteration $\ell = 0$, initialize $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$.
At iteration $\ell > 0$:
- Consumer n computes $\hat{\mathbf{x}}^n = \arg\max_{\mathcal{X}^n} U^n(\mathbf{x}^n) - p^n(\mathbf{x}^n)$, and communicates the set of $\hat{\mathbf{x}}^n$ to the LA.
- Supplier m computes $\hat{\mathbf{s}}^m = \arg\max_{\mathcal{S}^m} p^m(\mathbf{s}^m) - C^m(\mathbf{s}^m)$, and communicates the set of $\hat{\mathbf{s}}^m$ to the LA.
- The LA solves the restricted primal problem (14):
  If the objective value is 0, then the optimal solution is found.
  If the objective value is positive, the LA computes $p^n(\mathbf{x}^n) + \delta\bar{p}^n(\mathbf{x}^n)$, $p^m(\mathbf{s}^m) + \delta\bar{p}^m(\mathbf{s}^m)$, and communicates the updated prices $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$ to the consumers and suppliers.

In Algorithm BP, at each iteration, the consumers and suppliers receive bundle prices generated by the LA. Then, they submit to the LA the schedules that maximize their own welfare. Note that this is the opposite of what is being done in wholesale energy markets where participants submit bids and receive load and supply schedules. However, Algorithm BP is designed to be implemented internally in a microgrid, and it does not concern the LA's interaction with the wholesale market. Furthermore, an implicit assumption that is necessary for the proper operation of Algorithm BP is that the consumers and suppliers do not try to game the system and they truthfully report their schedules to the LA.

The following observations on Algorithm BP are immediate: First, the constraints (9) and (10) of the primal problem (7) are binding for only the set of schedules that are chosen by the consumers and suppliers. Therefore, the number of inequality constraints (15), (16) is equal to the number of these schedules. Furthermore, since these schedules are communicated to the LA by the system participants the LA does not need to keep track of the binding and non-binding constraints. Second, the market participants are required to solve their own welfare problems and they are assumed to truthfully report their schedules. It should be noted that these local problems can be hard to solve due to the arbitrary structure of the utility and cost functions. Nevertheless, the highly complex problem in (1) is decomposed into smaller problems that are more manageable by Algorithm BP. Third, the values of π, π_x^n, π_s^m, which require the knowledge of the utility and cost functions, are not needed by the mechanism. Instead, they are implicitly defined by the maximizations at the consumer and suppliers sides. Fourth, the update directions obtained via restricted primal and dual problems ensure that the algorithm converges to an optimal solution as stated in the following proposition.

Proposition 1. *Assume that the step size is chosen appropriately so that Lemma 1 and 2 hold. Then Algorithm BP* *converges to the optimal solution of problem (1).*

Proof. In Section II-B, it is already shown that problem (7) is equivalent to problem (1). Since (7) is a linear program, and update directions computed by Algorithm BP satisfy Lemma 1, it can be shown that the algorithm converges to the optimal solution of problem (7) using the same technique in [27]. □

A. The Step Size Choice

Note that the step size δ must be chosen by the LA *appropriately* for both Lemma 1 and Lemma 2 to hold. However, utility and cost functions are not known by the LA, and an appropriate choice of δ depends on the values of these functions. Next lemma provides a method for choosing the step size without the knowledge of the utility and cost functions under a mild assumption on these functions. In particular, we show that if the utility and cost functions take *integer* values, the LA can choose the step size using only the knowledge of the price values $p^n(\mathbf{x}^n)$ and $p^m(\mathbf{s}^m)$.

Before stating the lemma, we introduce the following definitions and notations. Denote the set of schedules chosen by consumer n with $\mathcal{X}_*^n$; the set of schedules chosen by supplier m with $\mathcal{S}_*^m$; the set of schedules that maximize $\sum_n p^n(\mathbf{x}^n) - \sum_m p^m(\mathbf{s}^m)$ with $\mathcal{F}_*$. Note that the complements of these sets are given by $\tilde{\mathcal{X}}_*^n = \mathcal{X}^n \backslash \mathcal{X}_*^n$, $\tilde{\mathcal{S}}_*^m = \mathcal{S}^m \backslash \mathcal{S}_*^m$, $\tilde{\mathcal{F}}_* = \mathcal{F} \backslash \mathcal{F}_*$. Let schedule variables with a star belong to the maximizing sets, and schedule variables with a tilde belong to the complements of the maximizing sets, i.e. for the schedules of consumer n we have $\mathbf{x}_*^n \in \mathcal{X}_*^n$ and $\tilde{\mathbf{x}}_*^n \in \tilde{\mathcal{X}}_*^n$. Define the following metrics:

$$\Delta_{\mathcal{X}}^n = \begin{cases} 1, & \text{if } p^n(\tilde{\mathbf{x}}_*^n) - p^n(\mathbf{x}_*^n) \text{ is integer,} \\ p^n(\tilde{\mathbf{x}}_*^n) - p^n(\mathbf{x}_*^n) - \lfloor p^n(\tilde{\mathbf{x}}_*^n) - p^n(\mathbf{x}_*^n) \rfloor, & \text{o.w.} \end{cases}$$

$$\Delta_{\mathcal{S}}^m = \begin{cases} 1, & \text{if } p^m(\tilde{\mathbf{s}}_*^m) - p^n(\mathbf{s}_*^m) \text{ is integer,} \\ p^m(\tilde{\mathbf{s}}_*^m) - p^m(\mathbf{s}_*^m) - \lfloor p^m(\tilde{\mathbf{s}}_*^m) - p^m(\mathbf{s}_*^m) \rfloor, & \text{o.w.} \end{cases}$$

Lemma 3. *Assume that the functions U^n and C^m take integer values for all n and m, respectively. Denote*

$$\Delta_{\mathcal{X},min}^n = \min_{\mathcal{X}_*^n, \tilde{\mathcal{X}}_*^n} \Delta_{\mathcal{X}}^n, \quad \Delta_{\mathcal{S},min}^m = \min_{\mathcal{S}_*^m, \tilde{\mathcal{S}}_*^m} \Delta_{\mathcal{S}}^m,$$

Choose $\delta_x^n > 0$ and $\delta_s^m > 0$ such that they are the largest real numbers that satisfy

$$\begin{aligned} \delta_x^n \left(\bar{p}^n(\tilde{\mathbf{x}}_*^n) - \bar{p}^n(\mathbf{x}_*^n)\right) &\geq -\Delta_{\mathcal{X},min}^n \\ \delta_s^m \left(\bar{p}^m(\tilde{\mathbf{s}}_*^m) - \bar{p}^m(\mathbf{s}_*^m)\right) &\geq -\Delta_{\mathcal{S},min}^m. \end{aligned} \tag{19}$$

Then, choosing $\delta = \min\{\min_n \delta_x^n,\ \min_m \delta_s^m, \delta_0\}$, where $0 < \delta_0 < \infty$, guarantees that Lemma 1 and Lemma 2 hold.

Proof. The proof for the consumer-side is given here. The proof for the supplier-side is similar.

First, note that $\pi_x^n = U^n(\mathbf{x}_*^n) - p^n(\mathbf{x}_*^n)$. Also, due to (15) $\bar{\pi}_x^n + \bar{p}^n(\mathbf{x}_*^n) \geq 0$. Combining these two together we have $\pi_x^n + \delta\bar{\pi}_x^n \geq U^n(\mathbf{x}_*^n) - p^n(\mathbf{x}_*^n) - \delta\bar{p}^n(\mathbf{x}_*^n)$ for all values of $\delta > 0$. Hence, both Lemma 1 and Lemma 2 hold for $\mathbf{x}_*^n \in \mathcal{X}_*^n$ regardless of the step size choice.

Now, consider the set $\tilde{\mathcal{X}}_*^n$. From the definition of $\Delta_{\mathcal{X}}^n$, and due to the assumption that U^n are integer valued we have

$$p^n(\tilde{\mathbf{x}}_*^n) - p^n(\mathbf{x}_*^n) - \Delta_{\mathcal{X}}^n \geq U^n(\tilde{\mathbf{x}}_*^n) - U^n(\mathbf{x}_*^n).$$

Rearranging the terms we obtain

$$\begin{aligned} U^n(\mathbf{x}_*^n) - p^n(\mathbf{x}_*^n) &\geq U^n(\tilde{\mathbf{x}}_*^n) - p^n(\tilde{\mathbf{x}}_*^n) + \Delta_{\mathcal{X}}^n \\ &\geq U^n(\tilde{\mathbf{x}}_*^n) - p^n(\tilde{\mathbf{x}}_*^n) + \Delta_{\mathcal{X},\min}^n \end{aligned} \tag{20}$$

We have

$$\begin{aligned} \pi_x^n + \delta\bar{\pi}_x^n &\geq U^n(\mathbf{x}_*^n) - p^n(\mathbf{x}_*^n) - \delta\bar{p}^n(\mathbf{x}_*^n) \\ &\geq U^n(\mathbf{x}_*^n) - p^n(\mathbf{x}_*^n) - \delta\bar{p}^n(\tilde{\mathbf{x}}_*^n) - \Delta_{\mathcal{X},\min}^n \\ &\geq U^n(\tilde{\mathbf{x}}_*^n) - p^n(\tilde{\mathbf{x}}_*^n) + \Delta_{\mathcal{X},\min}^n \\ &\quad - \delta\bar{p}^n(\tilde{\mathbf{x}}_*^n) - \Delta_{\mathcal{X},\min}^n \\ &= U^n(\tilde{\mathbf{x}}_*^n) - p^n(\tilde{\mathbf{x}}_*^n) - \delta\bar{p}^n(\tilde{\mathbf{x}}_*^n) \end{aligned} \tag{21}$$

where the first inequality follows from the above-proved fact that this relation holds for any value of the step size $\delta_x^n > 0$ for $\mathbf{x}_*^n \in \mathcal{X}_*^n$; second inequality follows from using (19) by noting that δ satisfies (19); third inequality follows from (20). This proves that both Lemma 1 and Lemma 2 hold, if the step size δ_x^n is chosen as this lemma suggests for schedules $\tilde{\mathbf{x}}_*^n \in \tilde{\mathcal{X}}_*^n$. Remembering that the lemmas hold regardless of the step size for $\mathbf{x}_*^n \in \mathcal{X}_*^n$, and noting that choosing the minimum of the step sizes among all consumers and suppliers preserve the above relationships, the lemmas hold for all n and m. Note that $\delta_x^n, \delta_s^m > 0$ and taking the minimum together with δ_0 assures that $0 < \delta < \infty$. □

B. Algorithm Implementation and Complexity

The welfare maximization problem of the LA (1) and its linear form over the feasible schedules sets (7) are difficult to solve due to the arbitrary structures of the utility and cost functions, and also due to the unavailability of their information at the LA. Algorithm BP, alleviates these difficulties by separating the problem into smaller ones that are solved independently by the consumers and suppliers via a distributed pricing mechanism. We note that the optimization problems solved by the consumers and suppliers are still difficult due to arbitrary utility and cost functions. However, these problems are much smaller compared to the LA's problem in (7) as they scale with the number of feasible schedules a user has. From a practical point of view, the number of feasible schedules need not be large. For example, a few schedules for different times of consumption during a day can be enough for an industrial load. Section IV demonstrates different flexibility models and corresponding consumption schedules in a typical scenario.

Another point worth mentioning is the LA's computational requirements. Note that in Algorithm BP, the LA should keep track of prices for all consumption and supply schedules. Although this is not a restrictive burden, identifying consumption-supply combinations that satisfy the constraint (8) has exponential complexity. Therefore, Algorithm BP is more appropriate for a system where the number of consumers is small, and the consumers are large entities with small number of schedules such as supermarkets, industrial loads, and data centers.

Finally, the convergence rate of Algorithm BP regarding the system size is of interest. Algorithm BP is derived using primal-dual decomposition methods. As it is usual with decomposition methods it takes hundreds or thousands of iterations for the algorithm to converge. Furthermore, problem (7) has $O(k(M+N))$ variables, and $O(k(M+N)+k^{M+N})$ constraints, where k is the number of feasible schedules for each user. Thus, it takes exponential time in the number of users to verify whether the constraints are binding or not. Yet, the difficulty of solving these problems is independent of the horizon length T, and provided that there are a small number of suppliers/consumers, the algorithm is expected to have a good performance.

Additionally, in settings with large number of suppliers/consumers, the formulations (and the performance of the algorithm) can be improved by eliminating "infeasible" schedules before the algorithm is run. For instance, if the load aggregator chooses schedules (by setting the appropriate λ variable to one) for which the induced supply in a given period cannot meet the induced demand in the same period, then this tuple of schedules are not feasible in (11). Thus, the corresponding λ variable can be dropped, and in the primal problem the associated constraint can be ignored. Identifying and pruning such infeasible schedules is an interesting algorithmic question, which would further improve the performance of the algorithm provided in our paper. This algorithmic question is a natural future direction to pursue.

Furthermore, the number of schedules (k) that the LA collects from suppliers and consumers can be viewed as a design variable. That is, if the algorithmic performance is a primary concern, the load aggregator can limit the number of different schedules that can be reported by the suppliers/consumers (thereby effectively limiting k). In light of the complexity of the underlying optimization problems provided above, this observation suggests that the LA can overcome the algorithmic difficulty by designing k appropriately.

IV. Numerical Results

In this section, we present numerical results to assess the performance of Algorithm BP. We consider a system that consists of 2 suppliers and 10 consumers. There are 9 *flexible* consumers and each consumer belongs to one of 3 different groups according to the type of flexibility they have in their demand. Note that there are 3 consumers in each flexibility type group. Also, there is another consumer whose demand is *inflexible*. This consumer is interested only in a given fixed schedule, or not receiving service at all, i.e., the consumer has two feasible schedules, one of which is the all-0 schedule. Flexible consumers of type 1, 2, and 3 have 2, 10, and 6 feasible schedules, respectively. On the other hand, the first supplier's cost is composed of linear cost of production and cost of ramping, i,e, $\sum_t c_p s(t) + c_r|s(t+1)-s(t)|$. This cost function introduces temporal dependency and does not bear *simple* analytic properties such as differentiability and convexity in its variable s. The second supplier's cost is chosen to be quadratic in the supply amount, i.e. $\sum_t c_t s(t)^2$. Furthermore, the values of cost functions are rounded to the nearest integer in order to implement the algorithm in the distributed way. Both suppliers have $2^3 \times 10^3 \times 6^3$ feasible schedules so that for every load combination there is a feasible supply schedule. The algorithm is run on a Windows computer with Intel Core i7-2600 CPU at 3.4 GHz and 16GB of ram. With the given size of the system, it took approximately 4 minutes for Algorithm BP to converge to the optimal solution.

In Figure 1, feasible load schedules of one consumer from each of the three flexibility groups are plotted. Consumer 1 has demand which can be served intermittently with regular periods throughout the day according to 2 feasible schedules. This type of consumer and demand can represent the air-conditioning demand of a supermarket or a data-center. On the other hand, consumer 2 prefers all of its demand to be served at a particular period of the day. The flexibility it bears is the amount of load that can be adjusted. Consumer 3 has constant amount of load that can be shifted in time. This demand profile exemplifies a factory task that requires significant amount of energy and that should be carried out one time a day, such as metal welding. We note that the utilities corresponding to the plotted consumption schedules are chosen to be arbitrary mappings from the schedule sets to real numbers.

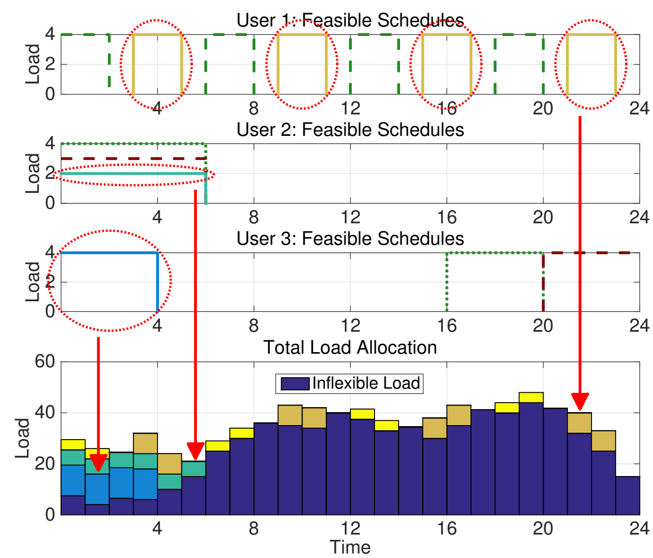

Fig. 1. Feasible schedules of 3 users and optimum total load allocation achieved by Algorithm BP.

At the bottom of Figure 1, the optimum load allocation achieved by Algorithm BP is plotted. The consumption schedules chosen by consumers and their placements over the time slots are highlighted by dotted ellipses and arrows. We observe that the algorithm actually exploits the flexibilities in consumers' demand and exhibits a *waterfilling* behavior. In particular, the schedules that fit the valleys of the inflexible load are chosen by the flexible consumers so that a flat aggregate load is obtained.

In Figure 2, prices computed by Algorithm BP for the feasible schedules of user 3 are plotted. Note that consumers make their load allocation decisions so that they maximize their own welfare which is defined as the utility obtained minus the payment made. In the figure, we observe that Algorithm BP assigns the lowest price value to the schedule of user 3 that is located at the lowest inflexible load periods.

Consequently, user 3 chooses its schedule which is assigned the lowest price (the utilities corresponding to this user's schedules are chosen to be close to each other, so the price is the determining factor). Thus, Algorithm BP successfully incentivizes the consumers to choose the schedules that fit the valleys of the inflexible load. As a result, decentralized decisions of the consumers are guided through Algorithm BP to maximize welfare of the whole system.

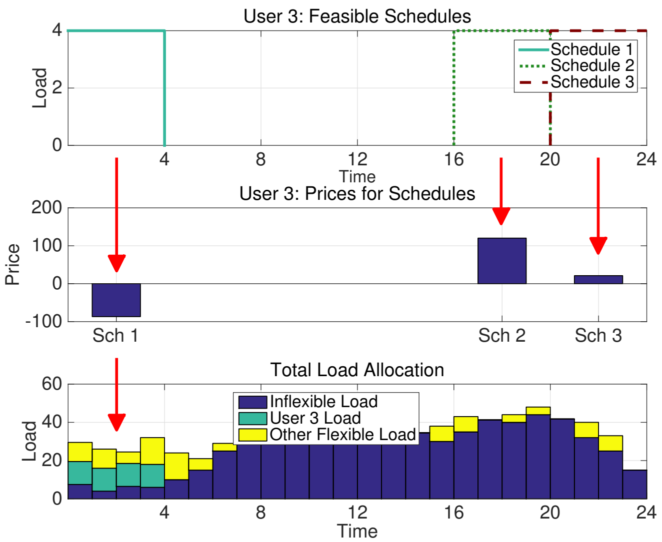

Fig. 2. Prices and allocations obtained by Algorithm BP for the feasible schedules of user 3.

In order to get a comparison for Algorithm BP, we simulated a simple pricing strategy for the LA. In this setup, the number and type of flexible users and their characteristics are kept the same as the previous simulations. Differently, there is one supplier whose cost function is given by $\sum_t \frac{1}{4}s(t)^2$. We assume that the LA has the knowledge of the supplier's cost function. In this setup, the LA determines *hourly* prices by setting them to the marginal cost of supply at inflexible load levels at each time slot. Therefore, the hourly prices reflect the system load levels. Then, given these prices flexible users determine their schedules that maximize their welfare as they do in Algorithm BP. It is also worth noting that negative prices can appear in the solution of the problem as they also can in dynamic electricity markets. These negative prices enable the creation of extra incentives for the flexible users to shift their load to more preferred slots.

The resulting load allocation determined by the described strategy and also the load allocation achieved by Algorithm BP are plotted in Figure 3. Note that the allocation achieved by Algorithm BP results in optimal welfare. In Figure 3, we observe that the allocations achieved by the two algorithms are different.

Since Algorithm BP achieves the optimal allocation, the comparison strategy does not achieve optimal welfare. Although, the comparison strategy does a good job in coordinating the flexible demand to be allocated at the time slots where inflexible load is lower (hence the marginal prices are lower), it fails to achieve a more balanced allocation as Algorithm BP does. In particular, the optimal BP algorithm

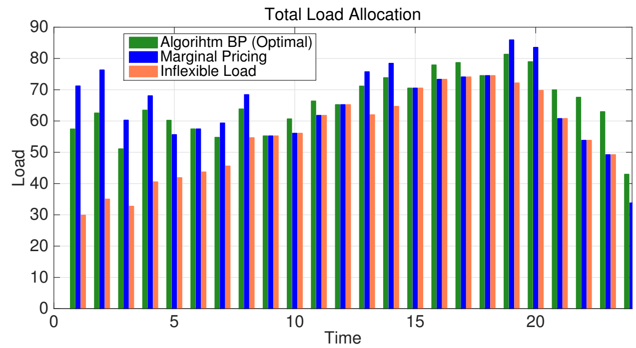

Fig. 3. Comparison of total load allocation. Marginal prices are set by using a convex supply cost function on inflexible load.

exhibits a dual improvement: (i) it yields higher user utilities due to its optimal utilization of specific user flexibilities for allocation; and (ii) it yields lower supply costs due its more balanced allocation of load over time. Overall, even for the small setup in our investigations (a total of 9 flexible users), BP achieves $\approx 2.5\%$ improvement in global welfare over the marginal allocation. These gains can be expected to increase as the diversity and the amount of flexibilities increase. Similarly, assume that the cost function of the supplier is modified to allow for increased marginal cost (e.g., by replacing the coefficient $1/4$ with a larger coefficient), or ramp constraints which are practically important. In this case, the flexibility of shifting load using prices obtained from Algorithm BP helps decrease costs, thereby further improving welfare.

V. Conclusion

In this work, we investigated the dayahead load and supply allocation problem in a microgrid. We did not assume any structure for demand and supply sides' operational and economic constraints other than restricting the users to have finite number of schedules to choose from. For example, consumer utility and supplier cost functions can be arbitrary and they do not have to have convexity or linearity properties. Furthermore, we let the consumer demand to be flexible in its amount, service time, and duration. Thus, our model can cover heterogeneous consumer and supplier bases in the same problem formulation.

Under the described system model, we formulate a social welfare maximization problem. Then, we derive a distributed primal-dual pricing algorithm. Under our algorithm, consumers and suppliers independently choose the schedules that fit best to their operational and economic interests. However, the prices generated by the algorithm coordinate the system participants in such a way that the whole system performance is improved. In particular, our algorithm achieves the optimal load and supply allocation. We demonstrate the dynamics of our algorithm via numerical results, and show that consumers and suppliers are incentivized to choose their schedules so that the overall system performance is optimized.

There are a number of directions in which this work can be extended and improved in future research. One such direction is the consideration of low-complexity variations of the bundle pricing approach that can yield close-to-optimal performance in large-scale operations. We have already noted in this paper several potential avenues of exploration in this direction. Another interesting direction of future research is the elimination of possible dishonest behavior including collusion of different parties (e.g., suppliers) in the economic interaction. While these can be imposed by enforcing legal laws in markets involving small number of suppliers and load aggregators, it would be of interest to release these assumptions in view of large markets involving many small parties, and therefore merits careful investigation.

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